

The local Calderòn problem and the determination at the boundary of the conductivity

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Abstract. We discuss the inverse problem of determining the, possibly anisotropic, conductivity of a body $\Omega \subset \mathbb{R}^n$ when the so-called Dirichlet-to-Neumann map is locally given on a non empty portion Γ of the boundary $\partial\Omega$. We extend results of uniqueness and stability at the boundary, obtained in [AG] where the Dirichlet-to-Neumann map was given on all of $\partial\Omega$ instead. We also obtain a pointwise stability result at the boundary among the class of conductivities which are continuous at some point $y \in \Gamma$. Our arguments also apply when the local Neumann-to-Dirichlet map is available.

1 Introduction.

In absence of internal sources, the electrostatic potential u in a conducting body, described by a domain $\Omega \subset \mathbb{R}^n$, is governed by the elliptic equation

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad (1.1)$$

where the symmetric, positive definite, matrix $\sigma = \sigma(x)$, $x \in \Omega$ represents the (possibly anisotropic) electric conductivity. The inverse conductivity problem consists of finding σ when the so called Dirichlet-to-Neumann (D-N) map

$$\Lambda_\sigma : u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega) \longrightarrow \sigma \nabla u \cdot \nu|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$$

is given for any $u \in H^1(\Omega)$ solution to (1.1). Here, ν denotes the unit outer normal to $\partial\Omega$. If measurements can be taken only on one portion Γ of $\partial\Omega$, then the relevant map is called the local Dirichlet-to-Neumann

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map. Let Γ be a non-empty open portion of $\partial\Omega$ and let us introduce the subspace of $H^{\frac{1}{2}}(\partial\Omega)$

$$H_{co}^{\frac{1}{2}}(\Gamma) = \{f \in H^{\frac{1}{2}}(\partial\Omega) \mid \text{supp } f \subset \Gamma\}. \quad (1.2)$$

The local Dirichlet-to-Neumann map is given, in a weak formulation, as the operator Λ_σ^Γ such that

$$\langle \Lambda_\sigma^\Gamma u, \phi \rangle = \int_\Omega \sigma \nabla u \cdot \nabla \phi, \quad (1.3)$$

for any $u, \phi \in H^1(\Omega)$, $u|_{\partial\Omega}, \phi|_{\partial\Omega} \in H_{co}^{\frac{1}{2}}(\Gamma)$ and u is a weak solution to (1.1).

The problem of recovering the conductivity of a body by taking measurements of voltage and current on its surface has come to be known as Electrical Impedance Tomography (EIT). Different materials display different electrical properties, so that a map of the conductivity $\sigma(x)$, $x \in \Omega$ (Ω domain in \mathbb{R}^n) can be used to investigate internal properties of Ω . EIT has many important applications in fields such as geophysics, medicine and non-destructive testing of materials. The first mathematical formulation of the inverse conductivity problem is due to A. P. Calderón [C], where he addressed the problem of whether it is possible to determine the (isotropic) conductivity by the D-N map. The case when measurements can be taken all over the boundary has been studied extensively in the past and fundamental papers like [A], [KV1], [KV2] and [SU] show that the isotropic case can be considered solved. On the other hand the anisotropic case is still open and different lines of research have been pursued. One direction has been to find the conductivity up to a diffeomorphism which keeps the boundary fixed (see [LU], [S], [N] and [LaU]).

Another direction has been the one to assume that the anisotropic conductivity is *a priori* known to depend on a restricted number of spatially-dependent parameters (see [A], [AG], [GL] and [L]).

The problem of recovering the conductivity σ by the knowledge of the local Dirichlet-to-Neumann map Λ_σ^Γ has been treated more recently. Lassas and Uhlmann in [LaU] recovered a connected compact real-analytic Riemannian manifold (M, g) with boundary by making use of the Green's function of the Laplace-Beltrami operator Δ_g . See also [LaUT].

The procedure of reconstructing the conductivity by local measurements has been studied first by Brown [B], where the author gives a formula for reconstructing the isotropic conductivity pointwise at the boundary of a Lipschitz domain Ω without any *a priori* smoothness assumption of the conductivity. Nakamura and Tanuma [NaT1] give a

formula for the pointwise reconstruction of a conductivity continuous at one point x^0 of the boundary from the local D-N map when the boundary is C^1 near x^0 . Under some additional regularity hypothesis the authors give a reconstruction formula for the normal derivatives of σ on $\partial\Omega$ at $x^0 \in \partial\Omega$ up to a certain order. A direct method for reconstructing the normal derivative of the conductivity from the local D-N map is presented in [NaT2]. The result in [NaT1] has been improved by Kang and Yun [KY] to an inductive reconstruction method by using only the value of σ at x^0 . The authors derive here also Hölder stability estimates for the inverse problem to identify Riemannian metrics (up to isometry) on the boundary via the local D-N map. An overview on reconstructing formulas of the conductivity and its normal derivative can be found in [NaT3].

For related uniqueness results in the case of local boundary data, we refer to Bukhgeim and Uhlmann [BU], Kenig, Sjöstrand and Uhlmann [KSU] and Isakov [I], and, for stability, Heck and Wang [HW]. Results of stability for cases of piecewise constant conductivities and local boundary maps have also been obtained by Alessandrini and Vessella [AV] and by Di Cristo [D].

It should also be emphasized that, mainly for the applications of medical imaging, and in particular for breast cancer detection by EIT, rather than the local Dirichlet-to-Neumann map, one should consider the so-called local Neumann-to-Dirichlet (N-D) map. That is, the map associating to specified current densities supported on a portion $\Gamma \subset \partial\Omega$ the corresponding boundary voltages, also measured on the same portion Γ of Ω .

In the present paper we study the inverse conductivity problem by local maps, concentrating on the issue of determining the boundary values of the conductivity and of its derivatives. We continue the line of investigation pursued in [AG], by considering anisotropic unknown conductivities having the structure $\sigma(x) = A(x, a(x))$, where $A(x, t)$ is a known, matrix valued, function and $a(x)$ is an unknown scalar function. The precise assumptions shall be illustrated in section 2. We improve upon the results obtained in [AG] under the following aspects.

- (i) The uniqueness and stability estimates are adapted to the local D-N map (Theorems 2.2, 2.4, 2.5 and Corollary 2.6),
- (ii) the stability estimate at the boundary is obtained in the wider class of conductivities which are continuous in a neighborhood of some point at the boundary (Theorem 2.3),
- (iii) analogous results are obtained when the local D-N map is replaced by the local N-D map (Theorem 2.7).

The paper is organized as follows. The main results are contained in section 2 (subsections 2.1, 2.2 for the local D-N, N-D maps respectively), while section 3 is devoted to the construction of singular solutions of equation (1.1) having the same type of singularity as those in [A] but having support compactly embedded on a non-empty open subset of the boundary (see Theorem 3.5). Proofs of the main results are given in section 4 (subsections 4.1, 4.2 for the local D-N, N-D maps respectively).

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2 Main results.

Let Ω be a domain in \mathbb{R}^n ($n \geq 3$), with Lipschitz boundary $\partial\Omega$. We recall, for sake of completeness, the definition of Lipschitz regularity of the boundary. We stick to notation already used in [AG].

DEFINITION 2.1. *Given positive numbers L, r, h satisfying $h \geq Lr$, we say that a bounded domain $\Omega \in \mathbb{R}^n$ has Lipschitz boundary if, for every $x^0 \in \partial\Omega$, there exists a rigid transformation of coordinates which maps x^0 into the origin, such that, setting $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, we have*

$$\begin{aligned} \Omega \cap \{x = (x', x_n) \mid |x'| < r, |x_n| < h\} = \\ = \{x = (x', x_n) \mid |x'| < r, |x_n| < h, x_n \geq f(x')\}, \end{aligned}$$

where $f = f(x')$ is a Lipschitz function defined for $|x'| < r$, which satisfies

$$\begin{aligned} f(0) &= 0 \\ |f(x') - f(y')| &\leq L |x' - y'|, \end{aligned}$$

for every $x', y' \in \mathbb{R}^{n-1}$, with $|x'|, |y'| < r$.

Let us now recall the class \mathcal{H} of functions $A(x, t)$ introduced in [AG], which will be considered as admissible conductivities.

DEFINITION 2.2. *Given $p > n$, the positive constants $\lambda, \mathcal{E}, \mathcal{F} > 0$, and denoting by Sym_n the class of $n \times n$ real valued symmetric matrices, we say that $A(\cdot, \cdot) \in \mathcal{H}$ if the following conditions hold*

$$A \in W^{1,p}(\Omega \times [\lambda^{-1}, \lambda], Sym_n), \quad (2.1)$$

$$D_t A \in W^{1,p}(\Omega \times [\lambda^{-1}, \lambda], Sym_n), \quad (2.2)$$

$$\begin{aligned} & \sup_{t \in [\lambda^{-1}, \lambda]} \left(\|A(\cdot, t)\|_{L^p(\Omega)} + \|D_x A(\cdot, t)\|_{L^p(\Omega)} \right. \\ & \left. + \|D_t A(\cdot, t)\|_{L^p(\Omega)} + \|D_t D_x A(\cdot, t)\|_{L^p(\Omega)} \right) \leq \mathcal{E}, \end{aligned} \quad (2.3)$$

$$\begin{aligned} \lambda^{-1} |\xi|^2 &\leq A(x, t) \xi \cdot \xi \leq \lambda |\xi|^2, & \text{for almost every } x \in \Omega, \\ & & \text{for every } t \in [\lambda^{-1}, \lambda], \xi \in \mathbb{R}^n. \end{aligned} \quad (2.4)$$

$$\begin{aligned} D_t A(x, t) \xi \cdot \xi &\geq \mathcal{F} |\xi|^2, & \text{for almost every } x \in \Omega, \\ & & \text{for every } t \in [\lambda^{-1}, \lambda], \xi \in \mathbb{R}^n. \end{aligned} \quad (2.5)$$

We observe that (2.4) is a condition of uniform ellipticity, whereas (2.5) is a condition of monotonicity with respect to the last variable t .

DEFINITION 2.3. *For every ρ , $0 < \rho < r$ we shall denote*

$$\Gamma_\rho = \{x \in \Gamma \mid \text{dist}(x, \partial\Gamma) > \rho\}, \quad (2.6)$$

$$U_\rho = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Gamma_\rho) < \frac{\rho}{4}\}, \quad (2.7)$$

$$U_\rho^i = U_\rho \cap \Omega. \quad (2.8)$$

Here it is understood that for the empty set \emptyset , we have $\text{dist}(x, \emptyset) = +\infty$. It is evident that, Γ being open and non-empty, there exists ρ_0 , $0 < \rho_0 \leq r$ such that Γ_{ρ_0} is also non empty. From now on we shall only consider values of ρ below ρ_0 .

Remark 2.1. *We emphasize that $\rho_0 > 0$ is a number which depends on the choice of $\Gamma \subset \partial\Omega$. It should be evident that if we choose Γ narrower and narrower, then ρ_0 tends to 0 and one should expect a deterioration in the stability estimates.*

2.1 The Dirichlet-to-Neumann map

We start by rigorously defining the local D-N map. We consider a given conductivity $\sigma \in L^\infty(\Omega, \text{Sym}_n)$ satisfying the ellipticity condition

$$\begin{aligned} \lambda^{-1}|\xi|^2 \leq \sigma(x)\xi \cdot \xi \leq \lambda|\xi|^2, \quad & \text{for almost every } x \in \Omega, \\ & \text{for every } \xi \in \mathbb{R}^n, \end{aligned} \quad (2.9)$$

and we fix an open, non-empty subset Γ of $\partial\Omega$. We denote by $\langle \cdot, \cdot \rangle$ the $L^2(\partial\Omega)$ -pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and its dual $H^{-\frac{1}{2}}(\partial\Omega)$.

DEFINITION 2.4. *The local Dirichlet-to-Neumann map associated to σ and Γ is the operator*

$$\Lambda_\sigma^\Gamma : H_{co}^{\frac{1}{2}}(\Gamma) \longrightarrow (H_{co}^{\frac{1}{2}}(\Gamma))^* \quad (2.10)$$

defined by

$$\langle \Lambda_\sigma^\Gamma g, \eta \rangle = \int_\Omega \sigma(x) \nabla u(x) \cdot \nabla \phi(x) dx, \quad (2.11)$$

for any $g, \eta \in H_{co}^{\frac{1}{2}}(\Gamma)$, where $u \in H^1(\Omega)$ is the weak solution to

$$\begin{cases} \operatorname{div}(\sigma(x)\nabla u(x)) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases}$$

and $\phi \in H^1(\Omega)$ is any function such that $\phi|_{\partial\Omega} = \eta$ in the trace sense.

Note that, by (2.11), it is easily verified that Λ_σ^Γ is selfadjoint. We shall denote by $\|\cdot\|_*$ the norm on the Banach space of bounded linear operators between $H_{co}^{\frac{1}{2}}(\Gamma)$ and $(H_{co}^{\frac{1}{2}}(\Gamma))^*$.

We can now state a first stability result for the boundary values of the conductivity.

THEOREM 2.2. (Lipschitz stability of boundary values). *Given $p > n$, let Ω be a bounded domain with Lipschitz boundary with constants L, r, h . Let Γ be the subset of $\partial\Omega$ introduced above and $\rho_0 = \rho_0(\Gamma)$ the positive number introduced in Definition 2.3. If a, b are two real-valued functions satisfying*

$$\lambda^{-1} \leq a(x), b(x) \leq \lambda, \quad \text{for every } x \in \Omega, \quad (2.12)$$

$$\|a\|_{W^{1,p}(\Omega)}, \|b\|_{W^{1,p}(\Omega)} \leq E, \quad (2.13)$$

for some positive constant $E > 0$ and $A \in \mathcal{H}$, then for any $\rho, 0 < \rho \leq \rho_0$

$$\|A(x, a(x)) - A(x, b(x))\|_{L^\infty(\bar{\Gamma}_\rho)} \leq C \|\Lambda_{A(x,a)}^\Gamma - \Lambda_{A(x,b)}^\Gamma\|_*. \quad (2.14)$$

Here $C > 0$ is a constant depending on $n, p, L, r, h, \operatorname{diam}(\Omega), \rho, \rho_0, \lambda, \mathcal{E}, \mathcal{F}$, but not on E .

The next Theorem improves upon the previous one, in that the regularity assumption (2.13) is relaxed to mere continuity.

THEOREM 2.3. (Pointwise stability at the boundary). *Given $p > n$, let Ω , Γ and ρ_0 be as in Theorem 2.2. Suppose a , b are two real valued functions satisfying (2.12) and furthermore are continuous on U_ρ^i , for some ρ , $0 < \rho \leq \rho_0$. Let $A \in \mathcal{H}$, then for any $x \in \Gamma_\rho$*

$$|A(x, a(x)) - A(x, b(x))| \leq C \| \Lambda_{A(x,a)}^\Gamma - \Lambda_{A(x,b)}^\Gamma \|_*, \quad (2.15)$$

where $C > 0$ is a constant which depends on n , p , L , r , h , $\text{diam}(\Omega)$, ρ_0 , ρ , λ , \mathcal{E} , \mathcal{F} only.

Here we state our stability results for boundary values of the derivatives of the conductivity.

THEOREM 2.4. (Hölder stability of derivatives at the boundary). *Given p , Ω , Γ and ρ_0 as in Theorem 2.1, let a , b satisfy (2.11), (2.12) and $A \in \mathcal{H}$. Suppose furthermore that for some ρ , $0 < \rho \leq \rho_0$, some positive integer k and some α , $0 < \alpha < 1$ we have*

$$A \in C^{k,\alpha}(\bar{U}_\rho \times [\lambda^{-1}, \lambda]), \quad (2.16)$$

$$\| A \|_{C^{k,\alpha}(\bar{U}_\rho \times [\lambda^{-1}, \lambda])} \leq E_k. \quad (2.17)$$

$$\| a - b \|_{C^{k,\alpha}(\bar{U}_\rho)} \leq E_k. \quad (2.18)$$

Then

$$\begin{aligned} & \| D^k(A(x, a(x)) - A(x, b(x))) \|_{L^\infty(\bar{\Gamma}_\rho)} \\ & \leq C \| \Lambda_{A(x,a)}^\Gamma - \Lambda_{A(x,b)}^\Gamma \|_*^{\delta_k \alpha}, \end{aligned} \quad (2.19)$$

where

$$\delta_k = \prod_{j=0}^k \frac{\alpha}{\alpha + j}. \quad (2.20)$$

Here $C > 0$ is a constant which depends only on n , p , L , r , h , $\text{diam}(\Omega)$, ρ_0 , ρ , λ , E , α , k , and E_k .

Under a slightly weaker assumption, we can also obtain the following uniqueness result.

THEOREM 2.5. (Uniqueness at the boundary). *Let $p, \Omega, \Gamma, \rho_0, a, b, A$ as in Theorem 2.3. Suppose that for some $\rho, 0 < \rho \leq \rho_0$ and some positive integer k we have*

$$a - b \in C^k(\bar{U}_\rho). \quad (2.21)$$

If

$$\Lambda_{A(x, a(x))}^\Gamma = \Lambda_{A(x, b(x))}^\Gamma,$$

then

$$D^j(a - b) = 0 \quad \text{on } \Gamma_\rho, \quad \text{for all } j \leq k. \quad (2.22)$$

If in addition we have

$$A \in C^k(\bar{U}_\rho \times [\lambda^{-1}, \lambda]), \quad (2.23)$$

then

$$D^j\left(A(x, a(x)) - A(x, b(x))\right) = 0 \quad \text{on } \Gamma_\rho, \quad \text{for all } j \leq k. \quad (2.24)$$

What follows is a well-known consequence of the previous Theorem, see [KV2] and [A] for related arguments.

COROLLARY 2.6. (Uniqueness in the interior). *Let a, b satisfy (2.11), (2.12) with $p = \infty$. Let $A \in \mathcal{H}$ and in addition $A \in W^{1, \infty}(\Omega \times [\lambda^{-1}, \lambda], \text{Sym}_n)$. Suppose that Ω can be partitioned into a finite number of Lipschitz domains, $\{A_j\}_{j=1, \dots, N}$, such that $a - b$ is analytic on each A_j .*

If

$$\Lambda_{A(x, a)}^\Gamma = \Lambda_{A(x, b)}^\Gamma,$$

then we have

$$A(x, a(x)) = A(x, b(x)) \quad \text{in } \Omega. \quad (2.25)$$

2.2 The Neumann-to-Dirichlet map

Let us introduce the following function spaces

$$\begin{aligned} {}_0H^{\frac{1}{2}}(\partial\Omega) &= \left\{ \phi \in H^{\frac{1}{2}}(\partial\Omega) \mid \int_{\partial\Omega} \phi = 0 \right\}, \\ {}_0H^{-\frac{1}{2}}(\partial\Omega) &= \left\{ \psi \in H^{-\frac{1}{2}}(\partial\Omega) \mid \langle \psi, 1 \rangle = 0 \right\}. \end{aligned}$$

Observe that if we consider the (global) D-N map Λ_σ , that is the map introduced in (2.10) Λ_σ^Γ in the special case when $\Gamma = \partial\Omega$, we have that,

it maps onto ${}_0H^{-\frac{1}{2}}(\partial\Omega)$, and, when restricted to ${}_0H^{\frac{1}{2}}(\partial\Omega)$, it is injective with bounded inverse. Then we can define the global Neumann-to-Dirichlet map as follows.

DEFINITION 2.5. *The Neumann-to-Dirichlet map associated to σ , $N_\sigma : {}_0H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow {}_0H^{\frac{1}{2}}(\partial\Omega)$ is given by*

$$N_\sigma = \left(\Lambda_\sigma|_{{}_0H^{\frac{1}{2}}(\partial\Omega)} \right)^{-1}. \quad (2.26)$$

Note that N_σ can also be characterized as the selfadjoint operator satisfying

$$\langle \psi, N_\sigma \psi \rangle = \int_\Omega \sigma(x) \nabla u(x) \cdot \nabla u(x) dx, \quad (2.27)$$

for every $\psi \in {}_0H^{-\frac{1}{2}}(\partial\Omega)$, where $u \in H^1(\Omega)$ is the weak solution to the Neumann problem

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0, & \text{in } \Omega, \\ \sigma \nabla u \cdot \nu|_{\partial\Omega} = \psi, & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u = 0. \end{cases} \quad (2.28)$$

We are now in position to introduce the local version of the N-D map. Let Γ be an open portion of $\partial\Omega$ and let $\Delta = \partial\Omega \setminus \bar{\Gamma}$. We denote by $H_{00}^{\frac{1}{2}}(\Delta)$ the closure in $H^{\frac{1}{2}}(\partial\Omega)$ of the space $H_{co}^{\frac{1}{2}}(\Delta)$ previously defined in (1.2) and we introduce

$${}_0H^{-\frac{1}{2}}(\Gamma) = \left\{ \psi \in {}_0H^{-\frac{1}{2}}(\partial\Omega) \mid \langle \psi, f \rangle = 0, \text{ for any } f \in H_{00}^{\frac{1}{2}}(\Delta) \right\}, \quad (2.29)$$

that is the space of distributions $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ which are supported in $\bar{\Gamma}$ and have zero average on $\partial\Omega$. The local N-D map is then defined as follows.

DEFINITION 2.6. *The local Neumann-to-Dirichlet map associated to σ , Γ is the operator $N_\sigma^\Gamma : {}_0H^{-\frac{1}{2}}(\Gamma) \longrightarrow ({}_0H^{-\frac{1}{2}}(\Gamma))^* \subset H^{\frac{1}{2}}(\partial\Omega)$ given by*

$$\langle N_\sigma^\Gamma i, j \rangle = \langle N_\sigma i, j \rangle, \quad (2.30)$$

for every $i, j \in {}_0H^{-\frac{1}{2}}(\Gamma)$.

When the local D-N map is replaced by the above defined local N-D map, completely analogous results to Theorems 2.2-2.5 and Corollary 2.6 could be obtained. For the sake of simplicity we state the appropriate version of Theorem 2.3 only. See also Remark 4.5 for further details. In what follows, we shall denote by $\|\cdot\|_{**}$ the norm on the Banach space of bounded linear operators between ${}_0H^{-\frac{1}{2}}(\Gamma)$ and $({}_0H^{-\frac{1}{2}}(\Gamma))^*$.

THEOREM 2.7. *Given $p > n$, let Ω , Γ and ρ_0 be as in Theorem 2.2. Suppose a , b are two real valued functions satisfying (2.12), continuous on U_ρ^i , for some ρ , $0 < \rho \leq \rho_0$. Let $A \in \mathcal{H}$, then for any $x \in \Gamma_\rho$*

$$|A(x, a(x)) - A(x, b(x))| \leq C \|N_{A(x, a)}^\Gamma - N_{A(x, b)}^\Gamma\|_{**}, \quad (2.31)$$

where $C > 0$ is a constant which depends on n , p , L , r , h , $\text{diam}(\Omega)$, ρ_0 , ρ , λ , \mathcal{E} , \mathcal{F} only.

3 Singular solutions vanishing on $\partial\Omega \setminus \Gamma$.

This section is devoted to the construction of particular solutions of equation (1.1), having the same type of singularity of those constructed in [A] but vanishing on the portion of the boundary $\partial\Omega \setminus \Gamma$. We consider the elliptic operator

$$L = \frac{\partial}{\partial x_i} \left(\sigma_{ij} \frac{\partial}{\partial x_j} \right), \quad \text{in } B_R = \{x \in \mathbb{R}^n \mid |x| < R\}, \quad (3.1)$$

where the coefficient matrix $(\sigma_{ij}(x))$ is symmetric and satisfies

$$\lambda^{-1} |\xi|^2 \leq \sigma_{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2, \quad \text{for every } x \in B_R, \xi \in \mathbb{R}^n, \quad (3.2)$$

and also

$$\|\sigma_{ij}\|_{W^{1,p}(B_R)} \leq E, \quad i, j = 1, \dots, n, \quad (3.3)$$

here $p > n$ and λ , E are positive constants. We recall the following theorem from [A].

THEOREM 3.1. (Singular Solutions). *Let L satisfy (3.1)-(3.3). For every spherical harmonic S_m of degree $m = 0, 1, 2, \dots$, there exists $u \in W_{loc}^{2,p}(B_R \setminus \{0\})$ such that*

$$Lu = 0 \quad \text{in } B_R \setminus \{0\}, \quad (3.4)$$

and furthermore,

$$\begin{aligned} u(x) &= \log |Jx| S_0 \left(\frac{Jx}{|Jx|} \right) + w(x), \text{ when } n = 2 \text{ and } m = 0, \\ u(x) &= |Jx|^{2-n-m} S_m \left(\frac{Jx}{|Jx|} \right) + w(x), \text{ otherwise,} \end{aligned} \quad (3.5)$$

where J is the positive definite symmetric matrix such that $J = \sqrt{(\sigma_{ij}(0))^{-1}}$ and w satisfies

$$|w(x)| + |x| |Dw(x)| \leq C |x|^{2-n-m+\alpha}, \quad \text{in } B_R \setminus \{0\}, \quad (3.6)$$

$$\left(\int_{s < |x| < 2s} |D^2 w|^p \right)^{\frac{1}{p}} \leq C s^{-n-m+\alpha+\frac{n}{p}}, \quad \text{for every } s, 0 < s < R/2. \quad (3.7)$$

Here α is any number such that $0 < \alpha < 1 - \frac{n}{p}$, and C is a constant depending only on α, n, p, R, λ , and E .

Proof. See [A, Theorem 1.1]. ■

We shall also need the following.

LEMMA 3.2. *Let the hypotheses of Theorem 3.1 be satisfied. For every $m = 0, 1, 2, \dots$ there exists a spherical harmonic S_m of degree m such that the solution u given by Theorem 3.1 also satisfies*

$$|Du(x)| > |x|^{1-(n+m)}, \quad \text{for every } x, \quad 0 < |x| < r_0, \quad (3.8)$$

where $r_0 > 0$ depends only on λ, E, p, m and R .

Proof. The proof of this lemma can be obtained along the same lines as of [A, Lemma 3.1] and [AG, Section 3]. ■

Let us construct now solutions u of (1.1) having a singularity of the same type of the above theorem in a point outside Ω and satisfying

$$u|_{\partial\Omega} \in H_{co}^{\frac{1}{2}}(\Gamma),$$

in the sense of traces. To this purpose we shall make use of an augmented domain Ω_ρ . In fact, for any $\rho, 0 < \rho \leq \rho_0$, one can always construct a domain Ω_ρ with Lipschitz constants depending only on ρ, r, L, h such that

$$\Omega \subset \Omega_\rho, \quad \partial\Omega \cap \Omega_\rho \subset\subset \Gamma \quad (3.9)$$

and

$$\text{dist}(x, \partial\Omega_\rho) \geq \frac{\rho}{2}, \quad \text{for every } x \in U_\rho. \quad (3.10)$$

If L is an operator of type (3.1) on Ω , satisfying (3.2), (3.3) on Ω , then for any $\rho > 0$, one can always extend the operator L to Ω_ρ in such a way so that L still satisfies (3.2), (3.3) on the enlarged domain Ω_ρ . As the boundary $\partial\Omega$ is Lipschitz the unit normal vector field to the boundary may not be defined pointwise so we shall introduce a unitary vector field

$\tilde{\nu}$ locally defined near $\partial\Omega$ such that: (i) $\tilde{\nu}$ is C^∞ smooth, (ii) $\tilde{\nu}$ is non-tangential to $\partial\Omega$ (see [AG, Section 3], for the construction procedure of the latter). The point $z_\tau = x^0 + \tau\tilde{\nu}$, where $x^0 \in \partial\Omega$, satisfies

$$C\tau \leq d(z_\tau, \partial\Omega) \leq \tau, \quad (3.11)$$

for any τ , $0 < \tau \leq \tau^0$. Here C and τ^0 are positive constants depending only on L , r , h [AG, Lemma 3.3].

We distinguish the cases when $m = 0$ or $m > 0$. For the case $m = 0$ we recall the following asymptotic estimate which only requires the Hölder continuity of the coefficients.

THEOREM 3.3. *Let Ω and Γ be as in Theorem 2.2. For any τ , $0 < \tau \leq \tau_0$, set $z_\tau = x^0 + \tau\tilde{\nu}$, for some $x^0 \in \bar{\Gamma}_\rho$ and ρ , $0 < \rho \leq \rho_0$. If L is the operator of (3.1), with Hölder continuous coefficients matrix $\sigma = \{\sigma_{ij}\}_{i,j=1\dots n}$, with exponent $0 < \beta < 1$, the Green's function G_σ for the Dirichlet boundary value problem*

$$\begin{cases} L G_\sigma(x, z_\tau) = -\delta(x - z_\tau), & \text{in } \Omega_\rho \\ G_\sigma(\cdot, z_\tau) = 0, & \text{on } \partial\Omega_\rho \end{cases}$$

has the form

$$G_\sigma(x, z_\tau) = C_n \left(\det(\sigma(z_\tau)) \right)^{-1/2} \left(\sigma^{-1}(z_\tau)(x - z_\tau) \cdot (x - z_\tau) \right)^{\frac{2-n}{2}} + R(x, z_\tau), \quad (3.12)$$

where C_n is a suitable dimensional constant and the remainder $R(x, z_\tau)$ satisfies

$$|R(x, z_\tau)| + |x - z_\tau| |\nabla_x R(x, z_\tau)| \leq C |x - z_\tau|^{2-n+\alpha}, \quad (3.13)$$

for every $x \in \Omega_\rho$, $|x - z_\tau| \leq r_0$, where $C = C(E)$ is a positive constant depending on E , r_0 is a positive number which depends only on the geometry of Ω and $0 < \alpha < \beta$.

Proof of Theorem 3.3. We refer to [Mi, Chapter 1] and [MT, (1.31)-(1.33)]. ■

As a Corollary, we also have

COROLLARY 3.4. *The Green's function G_σ introduced in Theorem 3.3 satisfies*

$$\|G_\sigma(\cdot, z_\tau)\|_{H^1(\Omega)} \leq C \tau^{(2-n)/2}, \quad \text{for any } 0 < \tau \leq \tau^0, \quad (3.14)$$

where $C > 0$ is a constant which only depends on $\text{diam}(\Omega)$, λ , L , r , h and τ^0 .

Proof of Corollary 3.4. A straightforward consequence of the point-wise upper bound (3.11) and of the Caccioppoli Inequality (see [Gi], Chapter 7 for example) yields

$$\|G_\sigma\|_{H^1(\Omega)} \leq \frac{K}{\tau} \|G_\sigma\|_{L^2(\Omega_\rho \setminus B_{C\frac{\tau}{2}}(z_\tau))}, \quad (3.15)$$

where $K = K(\lambda, L, r, h, \tau^0)$ is a positive constant depending only on λ, L, r, h and τ^0 and $C > 0$ in (3.15) is the constant introduced in (3.11). ■

For the case $m > 0$, we shall need stronger regularity assumptions on the coefficients. In fact, under the $W^{1,p}$ bound (3.3) we obtain

THEOREM 3.5. *Let Ω and Γ be as in Theorem 2.2 For any $\rho, 0 < \rho \leq \rho_0$, let z be an arbitrary point in U_ρ . For every $m = 0, 1, 2, \dots$ and for every spherical harmonic $S_m \neq 0$ of degree m , there exists $u \in H_{loc}^1(\overline{\Omega}_\rho \setminus \{z\}) \cap W_{loc}^{2,p}(\Omega_\rho \setminus \{z\})$ such that*

$$Lu = 0 \quad \text{in } \Omega_\rho \setminus \{z\}, \quad (3.16)$$

$$u = 0 \quad \text{on } \partial\Omega_\rho, \quad \text{in the trace sense} \quad (3.17)$$

and it has the form

$$u(x) = |J(x-z)|^{2-n-m} S_m \left(\frac{J(x-z)}{|J(x-z)|} \right) + v(x), \quad (3.18)$$

where J is the positive definite symmetric matrix such that $J = \sqrt{(\sigma_{ij}(z))^{-1}}$ and the remainder v satisfies

$$|v(x)| + |x-z| |Dv(x)| \leq C |x-z|^{2-n-m+\alpha}, \quad \text{in } B_{\rho/4}(z) \setminus \{z\}, \quad (3.19)$$

$$\left(\int_{s < |x-z| < 2s} |D^2 v|^p \right)^{\frac{1}{p}} \leq C s^{-n-m+\alpha+\frac{n}{p}}, \quad \text{for every } s, 0 < 2s < \rho/4. \quad (3.20)$$

Here α is any number such that $0 < \alpha < 1 - \frac{n}{p}$, and C is a constant depending only on $\alpha, n, p, R, \lambda, \rho_0, \rho$ and E .

Remark 3.6. Notice that, if $z \in U_\rho \setminus \Omega$ then $u \in H^1(\Omega)$ and its trace satisfies $u|_{\partial\Omega} \in H_{co}^{\frac{1}{2}}(\Gamma)$.

Proof of Theorem 3.5. With no loss of generality we can assume $z = 0$. Consider a positive number R sufficiently large so that $B_R(0)$, the ball

with centre 0 and radius R , is such that $\Omega_\rho \subset B_{R/2}(0) \subset\subset B_R(0)$. We consider the singular solution of Theorem 3.1 on $B_R(0)$. Let us denote this solution by u_m . Let w_0 be the solution to the problem

$$\begin{cases} \operatorname{div}(\sigma \nabla w_0) = 0, & \text{in } \Omega_\rho \\ w_0 = -u_m, & \text{on } \partial\Omega_\rho. \end{cases}$$

By recalling (3.5) we get

$$\sup_{\partial\Omega_\rho} (|u_m| + |\nabla u_m|) \leq C_1, \quad (3.21)$$

consequently

$$\|w_0\|_{H^1(\Omega_\rho)} \leq C_2, \quad (3.22)$$

where C_1 is a positive constant which depends on ρ_0, ρ, n and m only and $C_2 > 0$ depends only on $\rho_0, \rho, n, m, R, L, r$ and h . If we set

$$u(x) = |Jx|^{2-n-m} S_m \left(\frac{Jx}{|Jx|} \right) + v(x), \quad \text{for any } x \in \Omega_\rho \quad (3.23)$$

and $v = w + w_0$, where w is the reminder appearing in (3.5). Then u can be written as

$$u(x) = u_m(x) + w_0(x), \quad \text{for any } x \in \Omega_\rho \quad (3.24)$$

and satisfies (3.16), (3.17), moreover, by a standard interior regularity estimate

$$\|w_0\|_{W^{2,p}(B_{\rho/4}(z))} \leq C, \quad (3.25)$$

where $C > 0$ depends on ρ_0, ρ, m and n, R, L, r and h . Hence, recalling the bounds (3.6), (3.7) we obtain for $v = w + w_0$, (3.19), (3.20). ■

4 Proofs of the main theorems.

4.1 The D-N map.

The proofs of Theorem 2.4, 2.5 and Corollary 2.6 follow the same line of the corresponding results in [AG] by replacing the singular solutions used there by those introduced in the previous Section 3 which vanish on $\partial\Omega \setminus \Gamma$. For this reason, we shall give the details of the proof of Theorem 2.2 only.

Proof of Theorem 2.2. Let $x^0 \in \bar{\Gamma}_\rho$ such that $(a-b)(x^0) = \|a-b\|_{L^\infty(\Gamma_\rho)}$ and set $z_\tau = x^0 + \tau \tilde{\nu}$, with $0 < \tau \leq \min\{\tau_0, \frac{\rho}{8}\}$. Let G_a, G_b be the Green's

functions of Theorem 3.3 in Ω_ρ for the operators $\operatorname{div}(A(\cdot, a(\cdot))\nabla\cdot)$ and $\operatorname{div}(A(\cdot, b(\cdot))\nabla\cdot)$ respectively, that is, for instance

$$\begin{cases} \operatorname{div}(A(x, a(x))\nabla G_a(x, z_\tau)) = -\delta(x - z_\tau), & \text{in } \Omega_\rho \\ G_a(\cdot, z_\tau) = 0, & \text{on } \partial\Omega_\rho. \end{cases}$$

and analogously for G_b . By possibly reducing τ and taking $0 < \tau \leq \min\{\tau_0, \frac{\rho}{8}, \frac{r_0}{2}\}$, we have that $B_{r_0}(z_\tau) \cap \Omega$ is not empty and moreover $B_{r_0}(z_\tau) \cap \Omega \subset U_\rho$. By recalling (2.11) and [A, (b), p. 253] we can write

$$\begin{aligned} \int_{B_{r_0}(z_\tau) \cap \Omega} (A(x, a) - A(x, b)) \nabla G_a \cdot \nabla G_b + \int_{\Omega \setminus B_{r_0}(z_\tau)} (A(x, a) - A(x, b)) \nabla G_a \cdot \nabla G_b \\ = \langle (\Lambda_{A(x, a)}^\Gamma - \Lambda_{A(x, b)}^\Gamma) G_a, G_b \rangle. \end{aligned} \quad (4.1)$$

Here and in the sequel, it is understood $G_a = G_a(\cdot, z_\tau)$ and analogously for G_b . By combining (4.1) with (3.12) and (3.13), we obtain

$$\begin{aligned} & \frac{(2-n)^2}{(\det A(a, z_\tau))^{1/2} (\det A(b, z_\tau))^{1/2}} \\ & \times \int_{B_{r_0}(z_\tau) \cap \Omega} \frac{J_b^2(A(x, a) - A(x, b)) J_a^2(x - z_\tau) \cdot (x - z_\tau)}{(J_a^2(x - z_\tau) \cdot (x - z_\tau))^{n/2} (J_b^2(x - z_\tau) \cdot (x - z_\tau))^{n/2}} \\ & \leq C(E, \mathcal{E}) \left\{ \int_{B_{r_0}(z_\tau) \cap \Omega} |x - z_\tau|^{2(1-n)+\beta} \right. \\ & \quad + \int_{\Omega \setminus B_{r_0}(z_\tau)} |A(x, a) - A(x, b)| |x - z_\tau|^{2-2n} \Big\} \\ & \quad + \| \Lambda_{A(x, a)}^\Gamma - \Lambda_{A(x, b)}^\Gamma \|_* \| G_a \|_{H_{co}^{\frac{1}{2}}(\Gamma)} \| G_b \|_{H_{co}^{\frac{1}{2}}(\Gamma)}, \end{aligned} \quad (4.2)$$

with $J_a = \sqrt{(A(z_\tau, a(z_\tau)))^{-1}}$, $J_b = \sqrt{(A(z_\tau, b(z_\tau)))^{-1}}$ and by the Hölder continuity of $A(x, a(x))$, $A(x, b(x))$, (4.2) yields

$$\begin{aligned} & C(n, \mathcal{E}) \int_{B_{r_0}(z_\tau) \cap \Omega} \frac{(A(x^0, a(x^0))^{-1} - A(x^0, b(x^0))^{-1}) (x - z_\tau) \cdot (x - z_\tau)}{(J_a^2(x - z_\tau) \cdot (x - z_\tau))^{n/2} (J_b^2(x - z_\tau) \cdot (x - z_\tau))^{n/2}} \\ & \leq C(E, \mathcal{E}) \left\{ \int_{B_{r_0}(z_\tau) \cap \Omega} |x^0 - z_\tau|^\beta |x - z_\tau|^{2(1-n)} \right. \\ & \quad + \int_{B_{r_0}(z_\tau) \cap \Omega} |x - x^0|^\beta |x - z_\tau|^{2(1-n)} \\ & \quad + \int_{\Omega \setminus B_{r_0}(z_\tau)} |a - b| |x - z_\tau|^{2-2n} \Big\} \\ & \quad + \| \Lambda_{A(x, a)}^\Gamma - \Lambda_{A(x, b)}^\Gamma \|_* \| G_a \|_{H_{co}^{\frac{1}{2}}(\Gamma)} \| G_b \|_{H_{co}^{\frac{1}{2}}(\Gamma)}, \end{aligned} \quad (4.3)$$

where $C(n, \mathcal{E})$, $C(E, \mathcal{E})$ are positive constants depending only on n, \mathcal{E} and on E, \mathcal{E} respectively. Let us recall compute

$$\begin{aligned}
& (A(x^0, a(x^0))^{-1} - A(x^0, b(x^0))^{-1}) (x - z_\tau) \cdot (x - z_\tau) \\
&= \left(\int_{a(x^0)}^{b(x^0)} D_t(x^0, t)^{-1} dt \right) (x - z_\tau) \cdot (x - z_\tau) \\
&= \left(\int_{a(x^0)}^{b(x^0)} -A(x^0, t)^{-1} D_t(x^0, t) A(x^0, t)^{-1} dt \right) (x - z_\tau) \cdot (x - z_\tau) \\
&\geq \int_{a(x^0)}^{b(x^0)} \mathcal{F}^{-2} \lambda^{-2} |x - z_\tau|^2 dt, \tag{4.4}
\end{aligned}$$

where the ellipticity and the monotonicity assumptions (2.4), (2.5) had been used to obtain the lower bound estimate in (4.4). By recalling (4.16) and combining (4.3) with (4.4), we finally obtain

$$\begin{aligned}
C(n, \mathcal{E}) \mathcal{F}^{-2} \lambda^{-2} \|a - b\|_{L^\infty(\bar{\Gamma}_\rho)} \tau^{2-n} &\leq C(E, \mathcal{E}) \{ \tau^{2-n+\beta} + \tau^{2-n+\beta} + C_1 \} \\
&+ C_2 \| \Lambda_{A(x, a)}^\Gamma - \Lambda_{A(x, b)}^\Gamma \|_* \tau^{2-n},
\end{aligned}$$

where C_2 is a positive constant depending only on $\text{diam}(\Omega)$, λ , L , r , h and τ^0 . Consequently

$$\|a - b\|_{L^\infty(\bar{\Gamma}_\rho)} \leq C_2 f(\tau) + C_3 \| \Lambda_{A(x, a)}^\Gamma - \Lambda_{A(x, b)}^\Gamma \|_*,$$

where $C_2 > 0$ is a constant depending only on $n, \lambda, E, \mathcal{E}$ and \mathcal{F} , $C_3 > 0$ is a constant depending only on $n, \lambda, \mathcal{E}, \mathcal{F}, \text{diam}(\Omega), L, r, h$ and τ^0 and $f(\tau) \rightarrow 0$ as $\tau \rightarrow 0$. If we let $\tau \rightarrow 0$ we obtain (2.14). ■

We shall need three technical lemmas before we proceed with the proof of Theorem 2.3.

Given $a \in L^\infty(\Omega)$ satisfying the ellipticity condition (2.9) and such that it is continuous in \bar{U}_ρ^i we can extend a to all of \mathbb{R}^n in such a way that the ellipticity conditions are preserved and a is uniformly continuous in U_ρ . We shall continue to call a such an extended function. Let us denote by ω the modulus of continuity of a in U_ρ that is

$$|a(x) - a(y)| \leq \omega(|x - y|), \quad \text{for any } x, y \in \bar{U}_\rho, \tag{4.5}$$

where ω is a non negative real-valued function on \mathbb{R}^+ so that $\omega(t) \rightarrow 0$ as $t \rightarrow 0^+$. Let ϕ_ε , $\varepsilon > 0$, be a usual family of mollifying kernels with $\text{supp} \phi_\varepsilon \subset B_\varepsilon(0)$. We introduce the mollification of a as

$$a_\varepsilon = \phi_\varepsilon * a(x).$$

LEMMA 4.1. *For any $\varepsilon \leq \rho/2$ we have*

$$|a_\varepsilon(x) - a(x)| \leq \omega(\varepsilon), \quad \text{for any } x \in U_{\rho/2}, \quad (4.6)$$

where a_ε is the mollified function of step $\varepsilon > 0$.

Proof of Lemma 4.1. We have, for every $x \in \mathbb{R}^n$,

$$a_\varepsilon(x) - a(x) = \int_{|y-x| \leq \varepsilon} \phi_\varepsilon(x-y) (a(y) - a(x)) dy, \quad (4.7)$$

and when $x \in U_{\rho/2}$, $\varepsilon \leq \rho/2$, $|y-x| \leq \varepsilon$ implies $y \in U_\rho$, hence

$$\begin{aligned} |a_\varepsilon(x) - a(x)| &\leq \int_{|y-x| \leq \varepsilon} |\phi_\varepsilon(x-y)| \omega(\varepsilon) dy \\ &= \omega(\varepsilon), \quad \text{for any } x \in U_{\rho/2}. \quad \blacksquare \end{aligned}$$

LEMMA 4.2. *With the same assumptions as above*

$$\| \Lambda_{A(\cdot, a_\varepsilon(\cdot))}^\Gamma - \Lambda_{A(\cdot, a(\cdot))}^\Gamma \|_* \longrightarrow 0, \quad \text{as } \varepsilon \rightarrow 0^+, \quad (4.8)$$

Proof of Lemma 4.2. Let $\phi \in H_{co}^{\frac{1}{2}}(\Gamma)$, $0 < \varepsilon \leq \rho/2$ and take $u, u_\varepsilon \in H^1(\Omega)$ solutions to the problems

$$\begin{cases} \operatorname{div}(A(x, a) \nabla u) = 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \operatorname{div}(A(x, a_\varepsilon) \nabla u_\varepsilon) = 0 & \text{in } \Omega \\ u_\varepsilon = \phi & \text{on } \partial\Omega \end{cases}$$

respectively, then by (2.11) (see [A, (b), p. 253]) we have

$$\begin{aligned} \langle (\Lambda_{A(\cdot, a_\varepsilon(\cdot))}^\Gamma - \Lambda_{A(\cdot, a(\cdot))}^\Gamma) \phi, \phi \rangle &= \int_{\Omega} (A(x, a_\varepsilon) - A(x, a)) \nabla u_\varepsilon \cdot \nabla u \\ &= \int_{U_{\rho/2}^i} (A(x, a_\varepsilon) - A(x, a)) \nabla u_\varepsilon \cdot \nabla u \\ &\quad + \int_{\Omega \setminus U_{\rho/2}^i} (A(x, a_\varepsilon) - A(x, a)) \nabla u_\varepsilon \cdot \nabla u \quad (4.9) \end{aligned}$$

and by combining the Hölder continuity of $A(x, t)$ with Lemma 4.1

$$\begin{aligned} \int_{U_{\rho/2}^i} (a_\varepsilon - a) \nabla u_\varepsilon \cdot \nabla u &\leq C(\mathcal{F}) \omega(\varepsilon) \| \nabla u_\varepsilon \|_{L^2(\Omega)} \| \nabla u \|_{L^2(\Omega)} \\ &\leq \tilde{C} C(\mathcal{F}) \omega(\varepsilon) \| \phi \|_{H_{co}^{\frac{1}{2}}(\Gamma)}^2, \quad (4.10) \end{aligned}$$

where $C(\mathcal{F})$ is a positive constant depending on the constant of regularity \mathcal{F} for $A(x, t)$ and \tilde{C} is a positive constant which does not depend on ε . For any real numbers p, q with $\frac{1}{p} + \frac{2}{q} = 1$

$$\begin{aligned} \int_{\Omega \setminus U_{\rho/2}^i} (A(x, a_\varepsilon(x)) - A(x, a(x))) \nabla u_\varepsilon \cdot \nabla u &\leq C(\mathcal{F}) \|a_\varepsilon - a\|_{L^p(\Omega)} \\ &\cdot \|\nabla u_\varepsilon\|_{L^q(\Omega \setminus U_{\rho/2}^i)} \|\nabla u\|_{L^q(\Omega \setminus U_{\rho/2}^i)} \end{aligned} \quad (4.11)$$

By Meyers' inequality [M] we have that there exists $q > 2$ such that

$$\|\nabla u\|_{L^q(\Omega \setminus U_\rho)} \leq C \|\nabla u\|_{L^2(\Omega)} \quad (4.12)$$

and the same holds for u_ε and combining (4.10)-(4.12) we obtain

$$\begin{aligned} \|\Lambda_{A(\cdot, a(\cdot))}^\Gamma - \Lambda_{A(\cdot, a_\varepsilon(\cdot))}^\Gamma\|_* &= \sup_{\phi \in H_{co}^{\frac{1}{2}}(\Gamma), \|\phi\|_{H_{co}^{\frac{1}{2}}(\Gamma)} = 1} \langle (\Lambda_{A(\cdot, a(\cdot))}^\Gamma - \Lambda_{A(\cdot, a_\varepsilon(\cdot))}^\Gamma) \phi, \phi \rangle \\ &\leq C(\omega(\varepsilon) + \|a_\varepsilon - a\|_{L^p(\Omega)}), \end{aligned}$$

where C is a positive constant independent from ε . The above inequality holds for any ε , $0 < \varepsilon \leq \rho/2$, which concludes the proof. ■

Proof of Theorem 2.3. Let $x \in \Gamma_\rho$ and take $0 < \varepsilon \leq \rho/2$. We can split the quantity $|A(x, a(x)) - A(x, b(x))|$ as follows

$$\begin{aligned} |A(x, a(x)) - A(x, b(x))| &\leq |A(x, a(x)) - A(x, a_\varepsilon(x))| \\ &+ |A(x, a_\varepsilon(x)) - A(x, b_\varepsilon(x))| \\ &+ |A(x, b_\varepsilon(x)) - A(x, b(x))| \end{aligned}$$

and by the Hölder continuity of $A(x, t)$, Lemma 4.1 and Theorem 2.2

$$|A(x, a(x)) - A(x, b(x))| \leq C(2\omega(\varepsilon) + \|\Lambda_{a_\varepsilon}^\Gamma - \Lambda_{b_\varepsilon}^\Gamma\|_*), \quad (4.13)$$

where C is a positive constant which does not depend on ε . By letting $\varepsilon \rightarrow 0^+$ and Lemma 4.2, we obtain the desired estimate. ■

Proofs of Theorems 2.4, 2.5, 2.6 follow the same line of proofs of Theorems 2.2, 2.3, 2.4 of [AG] by replacing the singular solutions of Theorem 3.1 with the singular solutions with compact support in Γ obtained in Theorem 3.5.

4.2 The N-D map.

The proof of Theorem 2.7 shall be based on the following construction of singular solutions suited for the (2.28) with local data. The following is well known.

THEOREM 4.3. *Let Ω and Γ be as in Theorem 2.2. For any τ , $0 < \tau \leq \tau_0$, set $z_\tau = x^0 + \tau \tilde{\nu}$, for some $x^0 \in \bar{\Gamma}_\rho$ and ρ , $0 < \rho \leq \rho_0$. If L is the operator of (3.1), with Hölder continuous coefficients matrix $\sigma = \{\sigma_{ij}\}_{i,j=1\dots n}$, with exponent $0 < \beta < 1$, the Neumann's function N_σ for the boundary value problem associated to the operator (3.1)*

$$\begin{cases} L N_\sigma(x, z_\tau) = -\delta(x - z_\tau), & \text{in } \Omega_\rho \\ \sigma \nabla N_\sigma(x, z_\tau) \cdot \nu = \frac{1}{|\partial\Omega_\rho|}, & \text{on } \partial\Omega_\rho \end{cases}$$

has the form

$$N_\sigma(x, z_\tau) = C_n (\det(\sigma(z_\tau)))^{-1/2} \left(\sigma^{-1}(z_\tau)(x - z_\tau) \cdot (x - z_\tau) \right)^{\frac{2-n}{2}} + R(x, z_\tau), \quad (4.14)$$

where C_n is a suitable dimensional constant and the remainder $R(x, z_\tau)$ satisfies

$$|R(x, z_\tau)| + |x - z_\tau| |\nabla_x R(x, z_\tau)| \leq C |x - z_\tau|^{2-n+\alpha}, \quad (4.15)$$

for every $x \in \Omega_\rho$, $|x - z_\tau| \leq r_0$, where $C = C(E)$ is a positive constant depending on E , r_0 is a positive number which depends only on the geometry of Ω and $0 < \alpha < \beta$. Moreover

$$\|N_\sigma(\cdot, z_\tau)\|_{H^1(\Omega)} \leq C \tau^{(2-n)/2}, \quad \text{for any } 0 < \tau \leq \tau^0, \quad (4.16)$$

where $C > 0$ is a constant which only depends on $\text{diam}(\Omega)$, λ , L , r , h and τ^0 .

Proof. See the proof of Corollary 3.4 and [Mi, Chapter 1]. ■

THEOREM 4.4. *Let Ω and Γ be as in Theorem 2.2. For any τ , $0 < \tau \leq \tau_0$, set $z_\tau = x^0 + \tau \tilde{\nu}$, for some $x^0 \in \bar{\Gamma}_\rho$ and ρ , $0 < \rho \leq \rho_0$. If σ is the matrix with entries $\{\sigma_{ij}\}_{i,j=1\dots n}$ in (3.1) and S is an open portion of $\partial\Omega_\rho \setminus \partial\Omega$ with positive distance from $\partial\Omega$, there exists $u \in H_{loc}^1(\bar{\Omega}_\rho \setminus z_\tau)$ solution to*

$$\begin{cases} Lu = -\delta(x - z_\tau), & \text{in } \Omega_\rho \\ \sigma \nabla u \cdot \nu = 0, & \text{on } \partial\Omega_\rho \\ \sigma \nabla u \cdot \nu = -\frac{1}{|S|}, & \text{on } S. \end{cases}$$

Moreover

$$\|u(\cdot, z_\tau)\|_{H^1(\Omega)} \leq C \tau^{(2-n)/2} + B, \quad \text{for any } 0 < \tau \leq \tau^0, \quad (4.17)$$

where $C, B > 0$ are constants which only depend on $\text{diam}(\Omega)$, λ , L , r , h and τ^0 .

Proof. Let $N_\sigma(\cdot, z_\tau)$ be the Neumann function for Ω_ρ

$$\begin{cases} \operatorname{div}(\sigma \nabla N_\sigma(x, z_\tau)) = -\delta(x - z_\tau), & \text{in } \Omega_\rho \\ \sigma \nabla N_\sigma(x, z_\tau) \cdot \nu = -\frac{1}{|\partial\Omega_\rho|}, & \text{on } \partial\Omega_\rho \end{cases}$$

and S be an open portion of $\partial\Omega_\rho \setminus \partial\Omega$ with positive distance from $\partial\Omega$. Set

$$u(x) = N_\sigma(x, z_\tau) + w(x), \quad \text{for any } x \in \Omega_\rho,$$

where $w \in H^1(\Omega_\rho)$ is the solution to

$$\begin{cases} \operatorname{div}(\sigma \nabla w) = 0, & \text{in } \Omega_\rho \\ \sigma \nabla w \cdot \nu = -\frac{1}{|\partial\Omega_\rho|}, & \text{on } \partial\Omega_\rho \setminus S \\ \sigma \nabla w \cdot \nu = -\frac{|\partial\Omega_\rho \setminus S|}{|S| |\partial\Omega_\rho|}, & \text{on } S. \end{cases}$$

u is a solution of the given boundary value problem and by Caccioppoli inequality it also satisfies (4.17). ■

Proof of Theorem 2.7. It suffices to follow the arguments of the proof of Theorem 2.3 by simply replacing the appropriate singular solutions. ■

Remark 4.5. The argument introduced in Theorem 4.4 also enables to construct singular solutions of the type of those introduced in Theorem 3.5 which however satisfy the zero Neumann condition on $\partial\Omega \setminus \Gamma$. By means of such singular solutions it is rather obvious how the proofs of the remaining Theorems 2.2, 2.5 and Corollary 2.6 can be adapted when the local Dirichlet-to-Neumann map is replaced by the local Neumann-to-Dirichlet.

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